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Applied Mathematics and Finance [and Discussion]

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Applied mathematics and finance

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My aim is to make some comments of a quite general nature about the relation between applied mathematics and finance, theoretical and practical. I shall begin with a brief description of a case in which 'technology transfer' from a quite different area of mathematics, the Stefan problem, was helpful with a financial problem, namely the Black–Scholes approach to an American option. I then discuss some more general issues about the role of this kind of mathematics in finance and suggest some possible avenues for future progress.

1. Stefan problems and American options

The Stefan problem is a model for the melting or solidification of a pure material by heat transfer. Because the solid/liquid interface is *a priori* unknown, we must solve a free boundary problem, which is inherently nonlinear. We see below that the determination of the early exercise boundary for an American derivative product is a version of the Stefan problem.

Suppose that the temperature in a one-dimensional material $-\infty < x < \infty$ is $u(x,t)$. Suppose too that the material changes phase from solid to liquid or vice versa at a (scaled) temperature $u = 0$, and that this process requires the release or uptake of a latent heat which, again by a suitable scaling, may be taken to be 1. Consider a situation in which solid at the melting temperature occupies the region $-\infty < x < s(t)$, with liquid for $s(t) < x < \infty$. In this case there is just one free boundary $x = s(t)$ separating solid from liquid. Then a simple dimensionless model for the evolution of the free boundary is

$$u_t = u_{xx},$$

for $s(t) < x < \infty$, modelling the flow of heat, with

$$u(s(t),t) = 0, \quad -u_x(s(t),t) = \mathrm{d}s/\mathrm{d}t$$

describing the facts that the phase-change temperature is $u = 0$ and that energy is conserved. (We write u_x for $\partial u/\partial x$ and so on.)

The Stefan problem has a vast literature (see Tarzia (1988) for a bibliography containing 2500 papers). Indeed it has acted as an important canonical problem that has both stimulated important theoretical research (for example, the analysis of weak solutions and their relation to classical solutions) and has had considerable practical and numerical implications; a large number of technologies rely on solidification processes. Similar remarks might be made about generalizations of the Stefan model, such as the phase field model (Caginalp 1990) or the alloy problem (Elliott & Ockendon 1982); in another context altogether, the Navier–Stokes equations have likewise focused attention from all areas of applied mathematics.

We now show how to transform the Stefan problem above into a problem that is closely related to the valuation problem for American options. The main obstacle is

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the lack of smoothness of the first derivative of u and we overcome this by integrating in time. We define a new variable $v(x,t)$ by the Baiocchi transformation (Elliott & Ockendon 1982)

$$v(x,t) = \int_{s^{-1}(x)}^{t} u(x,\tau)\,\mathrm{d}\tau,$$

where $s^{-1}(x)$ is the inverse of $s(t)$. It is then clear that $v_t = u$. It is straightforward to show that v satisfies

$$v_t = v_{xx} + 1 \quad \text{for} \quad x > s(t), \tag{1}$$

$$v = 0 \quad \text{for} \quad x < s(t), \tag{2}$$

and that

$$v, v_x \quad \text{are continuous.} \tag{3}$$

Furthermore, in the melting case, when $u \geqslant 0$ everywhere, we have the inequality

$$v \geqslant 0. \tag{4}$$

Lastly, the initial data for v, $v(x,0)$, can be calculated from that of u by using (1). Equations (1)–(4) may be combined into the linear complementarity problem:

$$v \geqslant 0, \quad v_t - v_{xx} - 1 \leqslant 0, \tag{5}$$

$$v(v_t - v_{xx} - 1) = 0, \tag{6}$$

with appropriate initial data.

It is a short step from the linear complementarity problem to a variational inequality formulation of the problem. Equations (5) and (6) are multiplied by a test function $\phi(x,t)$ from a suitable space whose most important property is that its members satisfy the constraint (4). After integration by parts, we have

$$\int \frac{\partial v}{\partial t}(\phi - v) + \frac{\partial v}{\partial x}\left(\frac{\partial \phi}{\partial x} - \frac{\partial v}{\partial x}\right)\mathrm{d}x \geqslant 0,$$

for all test functions. (For details, see, for example Elliott & Ockendon (1982).) This weak version of the problem is essentially equivalent to a minimization over the convex set of functions that satisfy the constraint (together with suitable regularity conditions). It lends itself well to proofs of existence and uniqueness and, since it has the great advantage that the free boundary is given implicitly and need not be tracked explicitly, it is well-suited to numerical methods; a good example of the latter is the projected systematic overrelaxation (SOR) method (Cryer 1971).

We now turn to American options. As described in Wilmott *et al.* (1993), these can also be formulated as linear complementarity problems; the general structure is as follows. Suppose that $V(S,t)$ is the value of an option or other contingent claim depending on an underlying asset price S (the framework extends naturally to contingent claims depending on more than one asset) and with pay-off $V_T(S)$. Then the usual Black–Scholes model leads to the linear complementarity problem

$$V(S,t) \geqslant V_T(S), \quad \mathscr{L}_{\mathrm{BS}} V \leqslant 0, \tag{7}$$

$$V \cdot \mathscr{L}_{\mathrm{BS}} V = 0, \tag{8}$$

with $V(S,T) = V_T(S)$ and where V and V_S are continuous. Here $\mathscr{L}_{\mathrm{BS}}$ is the Black–Scholes differential operator associated with the market model used, so that if the interest rate r is constant and S follows the geometric process

$$\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}Z, \quad \mathscr{L}_{\mathrm{BS}} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} - r.$$

The inequalities and equality in (7) and (8) can be interpreted financially: the first inequality is the arbitrage constraint that the option value must lie above the pay-off (otherwise early exercise would lead to an arbitrage profit), whereas the second says that the return from the Black–Scholes portfolio of one option and $-V_S$ shares is at most that from a risk-free deposit, and can be less in regions where early exercise is optimal. The equality says that either early exercise is optimal, in which case the option value is equal to that of the pay-off, or the option satisfies the Black–Scholes equation. Finally, the continuity of V and V_S can also be justified using arbitrage arguments related to the exercise strategy of the holder.

It is a straightforward matter to convert the linear complementarity form of the option problem into a variational inequality for the heat equation, using a logarithmic price scale (i.e. set $S = E\,\mathrm{e}^x$ where E is a suitable scale for the pay-off function, such as the strike price for a vanilla option) and a scaled version of the option price. We are then in a position to import both theoretical and numerical methods from the corresponding Stefan problem and we can use methods such as the projected SOR algorithm to perform rapid accurate calculations. (It is also interesting to note that the option problem can be thought of as an optimal stopping problem – stopping being equated with exercise – and this route too leads to a Stefan problem; see van Moerbeke (1976) and Duffie (1992).) I do not give further details here (see Dewynne *et al*. 1993 or Wilmott *et al*. 1993), but I wish to stress the effectiveness of the 'technology transfer' at both theoretical and practical levels. It is our contention that many other areas of applied mathematics are ripe for exploitation in this way.

2. The relationship between mathematics and finance

Let us now turn to some more general issues concerning the relationship between mathematics and finance. Let us begin by hypothesizing the existence of a subject called 'mathematical finance', stimulated by increasing technical sophistication in financial markets (with a concomitant need for models and, consequently, analysis and computation). What is the nature of the interaction of this subject with economic theory, economic practice and, in particular, mathematics? What, if any, future does it have? (One reason for wondering whether it has a future as a formal subject is the difficulty posed by the lure of large salaries and problems of confidentiality.)

There is no doubt that the interaction with theoretical and practical economics has been a spectacular success, and we do not pursue these topics further. Turning to the interaction with mathematics, it may be helpful to consider the following questions.

Is mathematical finance purely a collection of mathematical methods, to be applied routinely to standard problems? Or is there, on the other hand, any feedback into mathematics, in the form of new ideas or new problems requiring the development of new mathematics? One of the intentions of this meeting is to show that there is, and perhaps, to indicate where future developments will take place. This aspect is discussed further below.

With which areas of mathematics does the interaction take place? Evidently, the majority of the interaction is with theoretical and applied probability, although activity in the area of differential equations is growing. It is likely that the latter area will become increasingly prominent because it is especially well-suited to asymptotic and numerical methods; this point is exemplified above and in Dewynne *et al.* (this volume).

Given that the differential equations encountered in finance are well in the mainstream of mathematical research and teaching, why are they so little known outside the finance community? Since Black–Scholes, the premier American general applied mathematical review journal, *SIAM Review*, has published just four papers on the subject (Samuelson 1973; Malliaris 1981, 1983; Ellerman 1984), whereas *SIAM Journal on applied Mathematics* has published only one (Dewynne & Wilmott 1994). (Incidentally, SIAM stands for Society for Industrial and Applied Mathematics; finance is surely the world's largest industry and one of the most mathematical.)

Why are there so few undergraduate courses in mathematics departments, despite the central places occupied by brownian motion and the diffusion equation? Finance is a striking and novel application of these ideas and it is a popular career destination for students. One possible reason is that the subject is regarded as 'unsuitable'; another is that the sheer volume of jargon and terminology, both financial and mathematical, that has to be mastered is too much; a third is that an over-rigorous approach to the subject in the existing literature has been off-putting. No doubt some of these reasons will disappear as more finance research is carried out in mathematics departments.

3. Future directions

According to Duffie (1992, pp. xiii, xiv), 'the decade spanning roughly 1969–79 seems like a golden age of dynamic asset pricing theory', whereas 'the decade or so since 1979 has, with relatively few exceptions, been a mopping up operation'. Should we be pessimistic about the outlook for the subject? In this section I suggest some ways in which the subject can develop that may lead to a fruitful interaction with applied mathematics.

One source of new problems is the explosion of 'exotic' contracts on the market; each new version poses a challenge to the mathematical modeller. Fortunately many of the common contracts can be analysed with a framework that generalizes the Black–Scholes analysis (see Wilmott *et al.* 1993) and do not pose substantial difficulties. In the train of these new contracts comes a range of problems in the efficient and rapid numerical calculation of derivative security values, whether in a real-time environment or, as in the area of risk-evaluation, involving very large numbers of contracts. However, the extension to existing concepts is not large, and for new developments we must look further.

Much of the body of finance theory developed to date is essentially linear, characterized by linear partial differential equations or, in the case of American options, which are of course nonlinear because they are free boundary problems, by a linear complementarity framework. A somewhat analogous body of 'linear' theory has been developed and as exhaustively explored for the Stefan problem, but there has also been much recent activity influenced by the problems posed by nonlinear effects such as surface tension effects or models posed in terms of nonlinear diffusion equations. It may be that finance is on the verge of a similar change of emphasis, prompted by the fact that market behaviour still does not exactly agree with the predictions of current models.

On a macroscopic level, the modelling of transaction costs leads immediately to nonlinear problems, for example as shown in this volume by Davis & Clark, Dewynne *et al.* and Pliska & Selby. Beyond this, models of the interaction between market players are only just beginning to emerge; they inevitably lead to nonlinear models as in Föllmer (this volume) and Schonbucher (1993); the latter considers the effect on the asset price of the existence of a large number of traders who follow the Black–Scholes hedging strategy.

Finally, there is much to be done in the detailed modelling of market behaviour. Patterson (1993) says 'I think that many editors and referees of finance journals have shown a studied lack of interest in [the topic of nonlinear dynamics of markets]'. Much of finance theory is built on the efficient market hypothesis yet empirical evidence, for example, Ziemba (this volume), shows that inefficiencies can exist. The scope for mathematical modelling and analysis here is enormous; it ranges from the delicate analysis of financial time series to models of (possibly irrational) investor behaviour in markets that may be inefficient and/or illiquid. There is no doubt that the practical side of the subject, which is after all its *raison d'être*, will drive intense investigation into these matters. It will be very interesting to see to what extent this investigation involves mathematics and mathematicians.

I am grateful to Jeff Dewynne and John Ockendon for many helpful conversations.

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Discussion

R. Lacey (*Derivative Investment Advisers Ltd, London, U.K.*). Many concepts of financial mathematics have a simple discrete formulation and a rather more difficult continuous one. Traders are well versed in the discrete case, for example the binomial model, but few could derive or solve the continuous time Black–Scholes equation. It might make things easier if new mathematical ideas were presented in the discrete case with the continuous case constructed in an appendix. If current research were formulated discretely, more traders would participate in applied mathematics rather than consume selected parts of applicable mathematics as interpreted by quantitative analysts. This would enable mathematicians and traders to work together in creating new applied financial mathematics.

Rather than taking the intuitively clear discrete case, constructing the continuous formulation and then creating another discrete formulation for calculation, it would seem to make sense to eliminate the last two steps. In some cases the discrete combinatorial analysis is more difficult than the limiting continuous time case.

S. D. Howison. There are several reasons for considering the continuous-time approach as preferable. One is that directly derived discrete models are often more difficult to analyse (for example, from the point of view of stability and error analysis). Also, to many people, the differential equations that arise from the continuous-time limit are easier to understand intuitively (after all, they are essentially the diffusion equation). This has even more force when applied to the nonlinear models that will be increasing in importance in the future.

A. D. Wilkie (*Watsons, Reigate, U.K.*) Two other areas worthy of consideration are time series analysis and stable distributions. Although in the short term share prices and other variables may move apparently in a random walk, equivalent to a continuous gaussian diffusion process, for the much longer term it is desirable to use time series modelling to investigate the way in which investment variables, dividend yields, interest rates, and so on, vary jointly in the long run. There is also evidence that the distributions of changes in share prices and other variables are fat-tailed, both in the short and the long run. The only consistent distribution for both timescales is the stable paretian series, of which the normal distribution is one. But the mathematics of the general stable paretian distribution is difficult so it would be useful to learn more about it.

T. Leach (*Assured Asset Management plc, Isle of Man, U.K.*). The British regulators should be made aware of the use of derivatives as an essential component in obtaining market exposure to minimize transaction costs, build in flexibility, and separate market and currency risks of international investments.

M. A. H. Dempster (*University of Essex, U.K.*). The approach to valuing American options based on complementarity problems and their equivalent variational inequalities, due originally to Jaillet *et al.* (*Acta appl. Math.* **21**, 263–289 (1990)), is flexible, powerful and easily extended to nonlinear problems. Partial differential equation methods give an accurate numerical approximation of the entire value surface and hence may be used to price many options simultaneously and to compute trivially all the 'Greeks' for risk management through hedging portfolios.